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Unified approach to Miura, Bäcklund and Darboux Transformations for Nonlinear Partial Differential Equations

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Abstract

This paper is an attempt to present and discuss at some length the Singular Manifold Method. This Method is based upon the Painlevé Property systematically used as a tool for obtaining clear cut answers to almost all the questions related with Nonlinear Partial Differential Equations: Lax pairs, Miura, Bäcklund or Darboux Transformations as well as τ-functions, in a unified way. Besides to present the basics of the Method we exemplify this approach by applying it to four equations in $(1+1)$-dimensions. Two of them are related with the other two through Miura transformations that are also derived by using the Singular Manifold Method.

1 Introduction

1.1 Integrability and the Painlevé Property

The beginnings of the study of singularities in the complex plane for differential equations has always been attributed to Cauchy [7]. Cauchy's main idea was to consider local solutions on the complex plane and to use methods of analytical prolongation to obtain global solutions. For this procedure to work, a complete knowledge of the singularities of the equation and its location in the complex plane is required. In this sense, it is essential to distinguish between two types of singularity.

- **Fixed singularities:** Singularities determined by the coefficients of the equation and its location does not therefore depend on initial conditions.
- **Movable singularities:** Singularities whose location on the complex plane does indeed depend on the initial conditions.

The truly decisive step towards the elucidation of the relationship between the analytical structure of a system and its integrability is attributed to the Russian mathematician **Sofia Kovalevskaya** [24]. Her work focused on the study of the motion of a rigid solid with a fixed point from an analysis of the singularities of the solutions [24]. Kovalevskaya's work was completely new and also addressed to uniquely determine the parameter values for which the only movable singularities of the solutions on the complex plane were poles.

Although Kovalevskaya's work was apparently afflicted by a lack of followers, in the last decade of the nineteenth century some mathematicians focused their attention on the classification of ordinary differential equations (ODEs) on the basis of the type of singularity their solutions were able to exhibit.

It was the French mathematician **Paul Painlevé** [32] who, following the ideas of Fuchs, Kovalevskaya, Picard and others, completely classified first order equations and studied second order equations. In this last case, he found 50 types of second order equations whose only movable singularities were ordinary poles. This special analytical property now carries his name and in what follows will be referred to as the **Painlevé Property (PP)** [35]. Of these 50 types of equation, 44 can be integrated in terms of known functions (Riccati equations, Elliptic Functions, Linear Equations, etc) and the other six, in spite of having meromorphic solutions, do not have algebraic integrals that allows one to reduce the equation to quadratures. Today these are known as Painlevé Transcendents. The main contribution of Paul Painlevé lies in that he established the basis for a theory that, unlike what had been believed until then, would allow one *a priori*, by singularity analysis, to decide on the integrability of an equation without previously solving it.

Although there is no definitive proof of why singularity analysis for an equation turns out to be a test of integrability, some circumstances do seem to corroborate this. For example, it has been found that Painlevé Transcendents often appear in similarity reductions of equations with solitons [3], [25]. It is also intriguing to note that always a certain relationship seems to exist between equations with the PP and Isomonodromy Transformations of certain linear equations [14]. The validity of Painlevé's analysis as a suitable procedure for detecting integrability could be related to the combined study of Algebraic Geometry and Abelian Function theory [34]. Jacobi elliptic functions have associated with them a genus 2 Riemann surface (torus with a hole) and on this surface all algebraic curves are meromorphic functions. Although hyperelliptic functions cannot generally be parametrized in terms of meromorphic functions, Jacobi observed that certain combinations of hyperelliptic integrals do have meromorphic inverses (this, for example, is the case of the integrals obtained by Kovalevskaya in the fourth integrable case for the rigid solid) and these are called Abelian integrals. Simultaneous study of these integrals together with the associated Riemann surfaces could be crucial for establishing the final proof of a reliable test of integrability based upon the ideas first developed by Paul Painlevé and Sofia Kovalevskaya.

1.2 The Ablowitz, Ramani and Segur algorithm

Between 1955 and 1960 the Korteweg-de Vries equation reappeared in the work of Fermi-Ulam-Pasta [15] and in the context of plasma physics. Towards 1965, with Zabusky and Kruskal [39], the concept of soliton emerged for the first time. The Soliton was an entity describing solitary wave solutions interacting among themselves without any

change in shape except for a small change in its phase. With this discovery in mind the Inverse Scattering Technique (IST) was developed [3], initially allowing one to solve the KdV equation and then many integrable equations with soliton solutions. The incredible success obtained with the IST technique prompted to Ablowitz, Ramani and Segur [2] to develop an algorithm (similar to that used by Kovalevskaya in the rigid solid problem) to determine whether an ordinary differential equation had the PP. As mentioned above, an ODE is said to have the PP if the only movable singularities of its solutions are poles. This same property can be stated by saying that all solutions are singlevalued except in the fixed singularities of the coefficients. The ARS (Ablowitz, Ramani and Segur) algorithm is a method for determining the nature of the singularities of the solutions of an ODE on the basis of an analysis of their local properties.

Until now we have considered the study of singularities within the context of systems described mathematically by ODEs. In view of the results obtained with the ARS algorithm, it seems natural to ask ourselves whether some other method for dealing with differential equations in partial derivatives (PDEs) could in principle be available. In particular, to find a new version of the PP that can be applied directly to the study of integrability for PDEs could indeed be extremely useful. One important problem here is that the solutions of a PDE are functions of at least two independent variables and their analytical continuation is clearly more complicated than in the case of ordinary equations.

The study of similarity reductions for PDEs that can be solved by IST led Ablowitz, Ramani and Segur to formulate what has now become know as the **ARS conjecture**: "Every ordinary differential equation that can be obtained as the similarity reduction of a PDE solvable by IST has the PP up to a smooth change of variables". This conjecture provides a necessary condition for checking whether a PDE is integrable or not. Ablowitz, Ramani and Segur [2] and Mc Leod and Olver [29] have tested some weak versions of this conjecture. Such demonstrations are based on the fact that if a PDE can be completely integrated its solutions can be evaluated in terms of the Gel'fand-Levitan-Marchenko linear integral equation that appears in the IST.

The ARS conjecture can therefore be understood in the sense that if it is possible to reduce a PDE to an ODE that does not have the PP (even after a suitable transformation of variables) it may be concluded that the PDE is not integrable. An algorithmic procedure has recently been put forward for determining similarity reductions for PDEs. The essence of the procedure is the study of the Lie symmetries. To check that a PDE has the PP using the ARS conjecture one must find all the possible similarity reductions and check that all the resulting ODEs do have the PP even though one has to make transformations of variables. In this context the ARS conjecture is on the one hand tedious (owing to the huge number of reductions to ODEs shown by some equations) and on the other hand becomes less useful as the number of symmetries shown by the equation decreases. It is also not clear which transformations of variables are permitted when checking whether the corresponding ODE is of the Painlevé type. In particular for equations that do not have symmetries the ARS conjecture is quite useless as it is not possible to obtain similarity reductions from usual group-theory procedures.

The obvious limitations of this method suggest that it would be interesting to have available a direct method (analogue of the ARS algorithm for ODEs) that would allow one to decide whether the PDEs under study are integrable. In this way one is extending the definition of the PP to partial derivatives using the original idea of Painlevé and

developing an algorithmic method to determine whether the equations have this property (without the requirement of considering all their similarity reductions to ODEs) and hence to decide whether they are integrable or not.

1.3 The Weiss, Tabor and Carnevale algorithm

The main difference between analytical functions of one and several variables is that the singularities of the latter are not isolated. If $f(z_1,\dots,z_n)$ is an analytical function of n complex variables z_i $(i=1,\dots,n)$, the singularities of f are in manifolds of $(2n-2)$-dimensions. These manifolds are determined by conditions of the form

$$\chi(z_1,\dots,z_n)=0, \tag{1.1}$$

where χ is an analytical function in a neighborhood of the manifold defined by (1.1). When this manifold depends on the initial conditions it is called a **movable singularity manifold**. The existence of these singularity manifolds suggests the need for introducing the PP concept for PDEs. This leads to a procedure to check whether the equations have such a property in a way that makes possible to evaluate the question of integrability by a unified analysis of singularities for both ODEs and PDEs. This was the work carried out by Weiss, Tabor and Carnevale (WTC) [35]. According to these authors, we say that **a PDE has the Painlevé property (PP) if its solutions are singlevalued in a neighborhood of the manifold of movable singularities**.

The WTC method also allows one to successfully apply some of the techniques developed for integrable systems to systems that are not completely integrable. Certain restrictions may be imposed on χ or on the parameters of the equation such that the solutions thus obtained only have poles as movable singularities. In this case, the singularity manifold χ is no longer an arbitrary function and the equation is said to have the conditional PP [6] and hence is partially integrable.

1.4 The singular manifold method

Weiss [36], [37] introduced the singular manifold method (SMM) which is an efficient algorithmic method to find the typical properties of integrable systems. If a PDE has the PP we have seen that its solutions can be expressed as a Laurent series in the form

$$u=\sum_{j=0}^{\infty}u_j(z_1,\dots,z_n)[\chi(z_1,\dots,z_n)]^{j-a}. \tag{1.2}$$

It is possible in any case to truncate the expansion series at a certain term in order to obtain particular solutions of the equation. If the expansion is truncated at the constant term (understood as the one that goes with ϕ^0 where we use ϕ instead of χ for the manifold in the truncated expansion), expression (1.2) reduces to:

$$u=u_0\phi^{-a}+u_1\phi^{1-a}+\cdots+u_a. \tag{1.3}$$

It is interesting to note that some attempts have been made in order to truncate the Laurent series at higher orders [33]; however here we shall only consider truncation at the constant term. Substitution of (1.3) in the corresponding PDE leads to an overdetermined

system of equations for ϕ, u_j and their derivatives. The essential point is that the singularity manifold ϕ is no longer an arbitrary function but rather -as we now shall see- it must fulfill certain equations due to the truncation condition. That's why the truncation of the Painlevé series is the basis of a method called **Singular Manifold Method** (SMM) that has been proved to be extremely successful in studying nonlinear PDEs. Many of the properties of such equations can be obtained through the SMM. Let us summarize some of them.

- The truncation (1.3) of the Painlevé series has itself the meaning of an auto-Bäcklund transformation between two solutions of a PDE [37], [12].

- The Lax pair can be obtained through the Singular Manifold equations [36], [30].

- The relation between the singular manifold method and nonclassical Lie symmetries of the truncated solutions has been studied in [10].

- The method of Hirota [19], [18], [20] is known as a powerful procedure for generating multisoliton solution for PDEs. It essentially consists in bilinearizing the differential equation by an *ansatz* reminiscent of the Painlevé truncated expansion.
  The WTC method also provides an iterative procedure for generating solutions [17] from the Lax pair and from the corresponding auto-Bäcklund transformation, where the corresponding singularity manifold ϕ is determined in each step and after n steps the solution can be expressed in terms of the product ϕ_1, ϕ_2, $\ldots$, ϕ_n from which it is then possible to construct the Hirota τ function associated with the solution with n solitons. The relationship between singular manifold and Hirota's τ-functions [9], [11], [13], [16] has also been clearly established.

- The Darboux transformations of a PDE [28], [4], [5], [27] are also an important procedure to obtain solutions of PDEs. The connection between SMM and Darboux transformations has been explained in different references [11], [13].

1.5 Plan of the paper

After the previous glimpses of evidence in regard to the relationship between the PP and the integrability conditions various directions in the search for integrable PDE become evident. On the one hand the Painlevé test does identify integrable systems and on the other hand the Singular Manifold Method appears as a systematic technique for finding Bäcklund and Darboux transformations, Lax pairs, Soliton Solutions etc. Our point of view -as we shall show henceforth- is that the PP should be used not only as a test of integrability but also as a fruitful source of information of practically all the important features of the Non Linear Partial Differential Equations. This paper tries to go an step further in this direction, adding the Miura transformations to the above mentioned properties arising from the SMM.

Sections 2 and 4 deal with the application of the SMM to two equations in $(1+1)$-dimensions as the AKNS (Ablowitz-Kaup-Newell-Segur) [1] and non local Boussinesq equation NLBq [26], [38] equation. The Lax pair, Darboux transformations and Solitonic Solutions are thus fully obtained in these cases. This also shows that our analysis

becomes not only a conceptual piece of information but also an algorithmic tool that can be systematically used.

Sections 3 and 5 are devoted to the study of the ShG (sinh-Gordon) [3] and KS (Kaup system) [21] systems. The Miura transformations between these equations and AKNS and NLBq respectively are obtained by using the SMM. Bäcklund transformations for AKNS and NLBq are derived. The two component induced Lax pairs for ShG and KS are identified by the same procedure.

Section 6 is one of conclusions. Some lengthy and/or auxiliary calculations are relegated to Appendices A to E.

2 The AKNS equation in (1+1)-dimensions

The well known [1] AKNS equation in 1+1,

$$0 = M_{yxxx} + 4M_y M_{xx} + 8M_x M_{xy} \tag{2.1}$$

is a nice and easy example to start to show how the method works.

The PP for this equation means that all solutions of (2.1) can be written as a series of the form (see [35]):

$$M = \sum_{j=0}^{\infty} M_j \chi^{j-a}, \tag{2.2}$$

where a is the leading index and χ is an arbitrary function of x and t, depending on the initial data, that is usually called **singularity manifold**. M_j are analytical functions of t and x in the neighborhood of $\chi = 0$. It is a trivial exercise to substitute (2.2) into (2.1) and to check that if the leading index is $a = 1$, the series (2.2) satisfies (2.1) for any functional form of χ. It could be said that (2.1) has the PP.

2.1 Truncated expansion. Bäcklund transformations

The SMM is based upon the above defined PP. It requires the truncation of (2.2) at the constant level $j = a$. It means, for our equation (2.1), that the truncated solutions should be:

$$M' = M + \frac{\phi_x}{\phi}, \tag{2.3}$$

where we have called $M = M_1$ and M' to the truncated solution. We have used here ϕ for the singularity manifold, instead of χ, to emphasize that the truncation implies that the singularity manifold is no longer an arbitrary function but a function that is "singularized" by the fact that it should satisfy some definite equations as we will see later. We shall be calling this manifold ϕ **singular manifold** [36], [37] and the method based on the truncation of the Painlevé series is the so called Singular Manifold Method.

The substitution of the truncated expansion (2.3) into the equation (2.1) provides the following results (see Appendix A):

- M as well as M' should be solutions of (2.1). It means that (2.3) could be consided as an auto-Bäcklund transformation between two solutions M' and M of the same equation.

• The solution M can be written in terms of the singular manifold in the following way:

$$M_x = -\left(\frac{1}{4}\right)\left(v_x + \frac{v^2}{2} + 2\lambda\right), \tag{2.4}$$

$$M_y = \frac{1}{2}(-v_y + 2\lambda q). \tag{2.5}$$

The notation that we have used is [13]:

$$v = \frac{\phi_{xx}}{\phi_x}, \tag{2.6}$$

$$q = \frac{\phi_y}{\phi_x}, \tag{2.7}$$

and λ is an arbitrary constant that, as we will see, plays the role of the spectral parameter.

• **The singular manifold equations**. The equations that the truncation procedure implies for ϕ are:

$$s_y = 4\lambda q_x, \tag{2.8}$$

where s is the schwartzian derivative defined as:

$$s = v_x - \frac{v^2}{2}. \tag{2.9}$$

Furthermore the compatibility condition $\phi_{xxt} = \phi_{txx}$ between the definitions (2.6) and (2.7) requires:

$$v_y = (q_x + qv)_x \Longrightarrow s_y = q_{xxx} + 2sq_x + qs_x. \tag{2.10}$$

It is not difficult to prove that the singular manifold equations are nothing but the AKNS system once again. In fact with the change of variables:

$$s = 4p_x + 2\lambda, \tag{2.11}$$

$$q = \frac{p_y}{\lambda}. \tag{2.12}$$

(2.8) is trivially fulfilled and (2.10) is written as:

$$0 = p_{yxxx} + 4p_y p_{xx} + 8p_x p_{xy} \tag{2.13}$$

that is obviously the AKNS system.

2.2 Lax pairs

As we have seen above, the singular manifold equations, written in terms of v and w are:

$$v_{xy} - v v_y = 4\lambda q_x, \tag{2.14}$$

$$v_y = (q_x + qv)_x \tag{2.15}$$

that can be considered as a new system of nonlinear equations. If we apply the Painlevé analysis to this system, the leading terms are (using ψ for the singularity manifold).

$$v \sim v_0 \psi^a, \qquad q \sim q_0 \psi^b.$$

The substitution in (2.14-15) provides:

$$a = -1, \qquad b = -2, \qquad v_0 = 2\psi_x, \qquad q_0 = -\frac{1}{\lambda}\psi_x \psi_y.$$

These leading terms provide the key for the linearization of the truncated solutions (2.3-4). Actually if we substitute v for its dominant term.

$$v = 2\frac{\psi_x}{\psi} \Longrightarrow \phi_x = \psi^2. \tag{2.16}$$

In such a case (2.4) is:

$$0 = \psi_{xx} + (2M_x + \lambda)\psi \tag{2.17}$$

and from (2.16), (2.5) and (2.10) we obtain:

$$2\frac{\psi_y}{\psi} = q_x + qv = \frac{1}{2\lambda}(2M_{xy} + v_{xy} + 2vM_y + vv_y)$$

or

$$0 = 2\lambda\psi_y + M_{xy}\psi - 2M_y\psi_x. \tag{2.18}$$

(2.17) and (2.18) are precisely the Lax pair for AKNS. To summarize it is possible to say that **the Lax pair is nothing but the singular manifold equations in which the eigenfuctions are directly obtained from the singular manifold** through (2.16).

2.3 Darboux transformations

Following an idea of Konopelchenko and Stramp [23], we can consider the Lax pair itself as a pair of coupled nonlinear equations between M and ψ. Let us now explain how to proceed.

As far as M' is also a solution of (2.1) an associated singular manifold ϕ'_2 linked to an spectral parameter λ_2 can be defined just by defining

$$\phi'_{2x} = \psi_2'^2 \tag{2.19}$$

a Lax pair for M' can be written as

$$0 = \psi'_{2xx} + (2M'_x + \lambda_2)\psi'_2, \tag{2.20}$$

$$0 = 2\lambda_2\psi'_{2y} + M'_{xy}\psi'_2 - 2M'_y\psi'_{2x}, \tag{2.21}$$

where the notation means that ψ'_2 is an eigenfunction for M' with eigenvalue λ_2. If we call ϕ_1 and ϕ_2 two singular manifolds for M attached to spectral parameters λ_1 and λ_2 respectively the corresponding eigenfunctions are defined as

$$\phi_{1x} = \psi_1^2, \tag{2.22}$$

$$\phi_{2x} = \psi_2^2, \tag{2.23}$$

and the Lax pairs take the form

$$0 = \psi_{1xx} + (2M_x + \lambda_1)\psi_1, \tag{2.24}$$

$$0 = 2\lambda_1\psi_{1y} + M_{xy}\psi_1 - 2M_y\psi_{1x}, \tag{2.25}$$

$$0 = \psi_{2xx} + (2M_x + \lambda_2)\psi_2, \tag{2.26}$$

$$0 = 2\lambda_2\psi_{2y} + M_{xy}\psi_2 - 2M_y\psi_{2x}. \tag{2.27}$$

If we use the singular manifold ϕ_1 to construct the truncated Painlevé expansion

$$M' = M + \frac{\phi_{1x}}{\phi_1} \tag{2.28}$$

and we then look at (2.20-21) as a system of nonlinear coupled equations a similar expansion should be performed for ψ'_2. That is to say:

$$\psi'_2 = \psi_2 + \frac{\Theta}{\psi_1}. \tag{2.29}$$

The substitution of the truncated expansions (2.28-29) in (2.20-21) provides the functional form for Θ. The result is

$$\Theta = -\psi_1\Omega(\psi_1, \psi_2), \tag{2.30}$$

where

$$\Omega(\psi_1, \psi_2) = \left(\frac{1}{\lambda_1 - \lambda_2}\right)(\psi_1\psi_{2x} - \psi_2\psi_{1x}). \tag{2.31}$$

The expansions (2.28-29) can be considered as transformations that leave invariant the Lax pair (2.18-19). In this sense they are Darboux transformations. It should be pointed out that we use the singular manifold ϕ_1 to actually realize the transformation (2.28) but not the eigenfunction ψ_1 as it is usual [28] in the Darboux transformations. Nevertheless eigenfunctions and Singular Manifolds are trivially related through (2.22) and therefore: **With two eigenfunctions ψ_1 and ψ_2 for M, we can construct an eigenfunction ψ'_2 for the iterated solution** M'. That is why we call them Darboux transformations.

2.4 Hirota's function

Furthermore, (2.19) is a nonlinear equation that relates ϕ'_2 and ψ'_2. It means that the singular manifold ϕ'_2 itself could also be expanded in terms of ϕ_1

$$\phi'_2 = \phi_2 + \frac{\Delta}{\phi_1} \tag{2.32}$$

and by substituting this expansion in (2.19) we obtain:

$$\Delta = -[\Omega(\psi_1, \psi_2)]^2. \tag{2.33}$$

The procedure described above could be easily iterated. The singular manifold ϕ'_2 for M' can be used to construct a new solution

$$M'' = M' + \frac{\phi'_{2x}}{\phi'_2} \tag{2.34}$$

that combined with (2.28) can be written as:

$$M'' = M + \frac{\tau_{12x}}{\tau_{12}}, \tag{2.35}$$

where

$$\tau_{12} = \phi'_2 \phi_1 \tag{2.36}$$

and by using (2.32) and (2.33)

$$\tau_{12} = \phi_2 \phi_1 - [\Omega(\psi_1, \psi_2)]^2. \tag{2.37}$$

It is an interesting point to note that the function τ_{12} for the second iteration is not a Singular Manifold but it can be constructed from two Singular Manifolds of the first iteration. The SMM provides algorithmically, but sometimes considered just a clever ansantz [16], [18], the Hirota's bilinear method [19]. It also provides the way to construct solutions for the τ-function, as we will see in the next section.

2.5 Solitonic solutions

The easiest nontrivial solutions can be obtained from the seminal solution

$$M = a_0 y. \tag{2.38}$$

For this solution, exponential solutions of (2.24-27) are

$$\psi_i = \exp\left(k_i x - \frac{a_0}{k_i} y\right), \tag{2.39}$$

where

$$\lambda_i = -k_i^2 \tag{2.40}$$

and the corresponding manifolds are

$$\phi_i = \frac{1}{2k_i}(\alpha_i + \psi_i^2), \tag{2.41}$$

where α_i are arbitrary constants. The equation (2.31) provides:

$$\Omega(\psi_1, \psi_2) = \frac{1}{k_1 + k_2}\psi_1\psi_2 \tag{2.42}$$

and (2.37) also yields

$$\tau_{12} = \frac{1}{4k_1k_2}(\alpha_1 + \psi_1^2)(\alpha_2 + \psi_2^2) - \frac{\psi_1^2\psi_2^2}{(k_1 + k_2)^2}. \tag{2.43}$$

We can write the first and second iteration as:

$$M' = a_0 y + \frac{\phi_{1x}}{\phi_1}, \tag{2.44}$$

$$M'' = a_0 y + \frac{\tau_{12x}}{\tau_{12}}, \tag{2.45}$$

where

$$\phi_1 = \frac{\alpha_1}{2k_1}(1 + F_1), \tag{2.46}$$

$$\tau_{12} = \frac{1\alpha_1\alpha_2}{4k_1k_2}\left\{1 + F_1 + F_2 + A_{12}F_1F_2\right\}, \tag{2.47}$$

$$\alpha_i = \exp\left(2k_i x_{0i}\right),$$

$$F_i = \exp\left(2k_i\left(x - \frac{a_0}{k_i^2}y - x_{0i}\right)\right), \tag{2.48}$$

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{2.49}$$

(2.44) corresponds to the one-soliton solution (see Fig. 1) and (2.45) to the interaction of two solitons (see Fig. 2).

3 The sinh-Gordon equation. Miura Transformation to AKNS

We shall be analyzing in this section how the SMM and the Singular Manifold equations are able to provide also information about Miura transformations between nonlinear PDE's. In particular, this section is devoted to obtain the Miura map between AKNS and the sinh-Gordon equation.

The sinh-Gordon equation [3]

$$U_{xy} = \sinh 2U \tag{3.1}$$

can be written as the system

$$0 = u_{xy} + 2u\eta_y, \tag{3.2}$$

$$0 = \eta_x + u^2 \tag{3.3}$$

through the change

$$u = U_x, \tag{3.4}$$

$$\eta_y = -\cosh 2U. \tag{3.5}$$

This system has the same problem as the sine-Gordon equation [31]. It has two Painlevé branches. In fact if we write the solutions of (3.2-3.3) as a Painlevé series

$$u = \sum_{j=0}^{\infty} u_j \chi^{j-a},$$
$$\eta = \sum_{j=0}^{\infty} \eta_j \chi^{j-b}.$$

The leading indexes are $a = b = 1$ but the dominant terms are:

$$u_0 = \pm\chi_x, \tag{3.6}$$

$$\eta_0 = \chi_x. \tag{3.7}$$

The $\pm$ sign of u_0 means that there are two possibilities for the expansion. This may seem at first a problem when one attempts to apply the SMM since if we choose a definite sign in the expansion we will be loosing information about the equation. This problem has been discussed during the last years (see [9], [13], [31], [8]). In these papers a modification of the SMM appears to be necessary. We need in fact to deal with two Singular Manifolds altogether. Obviously this fact represents a non trivial complication for the calculations.

We present here the easiest form to work with these two Singular Manifolds. The idea is the following: The two branches (3.6-7) suggest the following set of changes for the functions:

$$u = m - \hat{m}, \tag{3.8}$$

$$\eta = m + \hat{m} \tag{3.9}$$

in such a way that m and $\hat{m}$ should have only a Painlevé branch. It is now necessary to look for the equations that m and $\hat{m}$ should satisfy. For this purpose we introduce in (3.2-3) the change (3.8-9). Adding and subtracting the result we obtain:

$$m_{xy} + 2(m - \hat{m})m_y = 0, \tag{3.10}$$

$$\hat{m}_{xy} - 2(m - \hat{m})\hat{m}_y = 0. \tag{3.11}$$

From these equations we also obtain the additional information that we discuss in the next Subsections.

3.1 Miura Transformation

The $\hat{m}$ can be obtained from (3.10) and substituted in (3.11). The result is that m obeys:

$$0 = 2m_y m_{xxy} + 8m_x m_y^2 - m_{xy}^2 \tag{3.12}$$

that is the integrated version of the AKNS equation

$$0 = m_{yxxx} + 4m_y m_{xx} + 8m_x m_{xy}. \tag{3.13}$$

In a similar form, we can obtain m from (3.11) and the substitution in (3.10) is

$$0 = 2\hat{m}_y \hat{m}_{xxy} + 8\hat{m}_x \hat{m}_y^2 - \hat{m}_{xy}^2 \tag{3.14}$$

that it is again the integration of the AKNS equation

$$0 = \hat{m}_{yxxx} + 4\hat{m}_y \hat{m}_{xx} + 8\hat{m}_x \hat{m}_{xy}. \tag{3.15}$$

As a consequence of what has just been said the change of functions (3.8-9) can be inverted to yield

$$2m_x = u_x + \eta_x = u_x - u^2, \tag{3.16}$$

$$2\hat{m}_x = \eta_x - u_x = -u_x - u^2 \tag{3.17}$$

which represents the Miura transformations between the sinh-Gordon system (3.2-3) and the AKNS equations (3.13) and (3.15).

3.2 Bäcklund Transformations

As an additional result we obtain that the solutions (u, η) of sinh-Gordon can be constructed by using two solutions m and $\hat{m}$ of AKNS. Nevertheless these solutions are not independent. They are related by (3.10-11) that can be written as:

$$\hat{m} = m + \frac{1}{2}\frac{m_{xy}}{m_y}, \qquad m = \hat{m} + \frac{1}{2}\frac{\hat{m}_{xy}}{\hat{m}_y}, \tag{3.18}$$

$$(m_y \hat{m}_y)_x = 0. \tag{3.19}$$

These equation (3.18-19) can easily be recognized as Bäcklund transformations between the two solutions m and $\hat{m}$ of AKNS.

3.3 Singular manifold method with two manifolds

From the above discussion it is easy to understand why it seems reasonable to talk about two Singular Manifolds, one for the expansion of m and the other for the expansion on $\hat{m}$. Let us call ϕ the singular manifold for m and $\hat{\phi}$ the singular manifold for $\hat{m}$. The truncated expansions are

$$m' = m + \frac{\phi_x}{\phi}, \tag{3.20}$$

$$\hat{m}' = \hat{m} + \frac{\hat{\phi}_x}{\hat{\phi}}, \tag{3.21}$$

and the corresponding expansions for u and η are:

$$u' = u + \frac{\phi_x}{\phi} - \frac{\hat{\phi}_x}{\hat{\phi}}, \tag{3.22}$$

$$\eta' = \eta + \frac{\phi_x}{\phi} + \frac{\hat{\phi}_x}{\hat{\phi}}. \tag{3.23}$$

However ϕ and $\hat{\phi}$ are not independent because m and $\hat{m}$ are related by the Bäcklund transformation (3.18-19). This is reminiscent of the requirement that u and η satisfy (3.2-3). In fact substituting (3.22-23) in (3.3) we obtain the coupling condition between ϕ and $\hat{\phi}$ (see Appendix B)

$$\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}} = A\frac{\phi_x}{\phi} + \hat{A}\frac{\hat{\phi}_x}{\hat{\phi}}, \tag{3.24}$$

where

$$A = \frac{v}{2} + u, \tag{3.25}$$

$$\hat{A} = \frac{\hat{v}}{2} - u. \tag{3.26}$$

The notation is the one defined in (2.6).

3.4 Lax pair for sinh-Gordon

The derivative of (3.24) with respect to x (see Appendix B) provides:

$$A_x = A(\hat{v} - A - \hat{A}) = \frac{\hat{v} - v}{2}, \tag{3.27}$$

$$\hat{A}_x = \hat{A}(v - A - \hat{A}) = \frac{v - \hat{v}}{2}. \tag{3.28}$$

We should remember at this point that ϕ and $\hat{\phi}$ are Singular Manifolds for AKNS and that the change

$$\phi_x = \psi^2, \qquad \hat{\phi}_x = \hat{\psi}^2 \tag{3.29}$$

relates the Singular Manifolds with the eigenfunctions of the Lax pair (2.17-18) of AKNS. By combining (3.29) with (3.27-28) we can integrate out the variable x, and finally obtain

$$A = a\frac{\hat{\psi}}{\psi}, \tag{3.30}$$

$$\hat{A} = \hat{a}\frac{\psi}{\hat{\psi}}, \tag{3.31}$$

where a and $\hat{a}$ are constants. (3.25-26) can be now written as:

$$\psi_x = a\hat{\psi} - u\psi, \tag{3.32}$$

$$\hat{\psi}_x = \hat{a}\psi + u\hat{\psi}, \tag{3.33}$$

where ψ and $\hat{\psi}$ are solutions of the Lax pair of AKNS. That is equivalent to

$$0 = \psi_{xx} + (2m_x + \lambda)\psi, \tag{3.34}$$

$$0 = 2\lambda\psi_y + m_{xy}\psi - 2m_y\psi_x, \tag{3.35}$$

$$0 = \hat{\psi}_{xx} + (2\hat{m}_x + \hat{\lambda})\hat{\psi}, \tag{3.36}$$

$$0 = 2\hat{\lambda}\hat{\psi}_y + \hat{m}_{xy}\hat{\psi} - 2\hat{m}_y\hat{\psi}_x. \tag{3.37}$$

The compatibility between (3.32-33) and (3.34), (3.36) requires

$$\lambda = \hat{\lambda} = -a\hat{a}. \tag{3.38}$$

The y component of the Lax pair is given by (3.35), (3.37) and can be written as

$$2\hat{a}\psi_y = -(u_y + \eta_y)\hat{\psi}, \tag{3.39}$$

$$2a\hat{\psi}_y = (u_y - \eta_y)\psi. \tag{3.40}$$

Therefore, the Lax pair for sinh-Gordon can be written in its usual matrix form as:

$$\begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}_x = \begin{pmatrix} -u & a \\ \hat{a} & u \end{pmatrix} \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}, \tag{3.41}$$

$$\begin{pmatrix} 2\hat{a}\psi \\ 2a\hat{\psi} \end{pmatrix}_y = \begin{pmatrix} 0 & u_y + \eta_y \\ u_y - \eta_y & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}. \tag{3.42}$$

Conclusion: The method that we derived in Section 2 to obtain Darboux transformations and solutions for AKNS can be directly applied to sinh-Gordon. The construction of solution of sinh-Gordon can be done through (3.8-9) by using two solutions of AKNS related by the Bäcklund transformation (3.18). From the point of view of the Singular Manifold this implies that ϕ and $\hat{\phi}$ are related by the coupling condition (3.24). Using (3.29-31) this condition can be written in a much easier form as:

$$a\hat{\phi} + \hat{a}\phi = \psi\hat{\psi}, \tag{3.43}$$

$$\hat{\lambda} = \lambda = -a\hat{a}. \tag{3.44}$$

We believe that the splitting (3.8-9) is the key to solve the long standing problem that concerns to the application of the SMM to equations with two branches. In the next sections we will return to the same topic in a different but not unrelated context.

4 Non local Boussinesq equation

The following system of equations

$$N_x = M_t, \tag{4.1}$$

$$M_x N_t = M_x M_{xxx} + 2M_x^3 + M_t^2 - M_{xx}^2 \tag{4.2}$$

or equivalently

$$M_x^2(M_{tt} - M_{xxxx}) = 4M_x^3 M_{xx} + 2M_x(M_t M_{tx} - M_{xx}M_{xxx}) - M_{xx}(M_t^2 - M_{xx}^2) \tag{4.3}$$

has been considered in [26], [38] as related with the Kaup (sometimes called Classical Bousinesq) system through a Miura transformation. The Kaup system is a good example of system with two branches [21], [9], [8]. Nevertheless (4.1-2) has only one branch. That is why we are interested in this section in studying the relationship of both equations from the point of view of the SMM and to relate the corresponding results with the Kaup system in Section 5.

4.1 Truncated expansion. Bäcklund transformations

The leading terms of (4.1-2) are

$$M \sim \chi_x \chi^{-1},$$

$$N \sim \chi_t \chi^{-1}$$

that suggests the truncated expansion

$$M' = M + \frac{\phi_x}{\phi} \longrightarrow N' = N + \frac{\phi_t}{\phi}. \tag{4.4}$$

From this equations we obtain the following set of results (see Appendix C):

- (4.3) could be consided as an auto-Bäcklund transformation between two solutions M' and M of the same equation.
- The solution M can be written in terms of the Singular Manifold as:

$$M_x = \left(\frac{1}{4}\right)\left((w+2\lambda)^2 - v^2\right), \tag{4.5}$$

$$M_t = \frac{1}{2}\left\{(w+2\lambda)v_x - v w_x + (w+\lambda)\left[(w+2\lambda)^2 - v^2\right]\right\}, \tag{4.6}$$

where [13]:

$$v = \frac{\phi_{xx}}{\phi_x}, \tag{4.7}$$

$$w = \frac{\phi_t}{\phi_x}, \tag{4.8}$$

and λ is an arbitrary constant.

- **The singular manifold equations** can be written as the system of PDEs:

$$v_t = (w_x + wv)_x \tag{4.9}$$

$$w_t = \left(v_x - \frac{v^2}{2} + \frac{3}{2}(w+2\lambda)^2 - 2\lambda(w+2\lambda)\right)_x. \tag{4.10}$$

Notice that the parameter λ can be removed from these equations through the galilean transformation

$$\bar{w} \to (w + 2\lambda),$$

$$\bar{x} \to x - 2\lambda t$$

that transforms (4.9-10) into

$$v_t = (\bar{w}_{\bar{x}} + \bar{w}v)_{\bar{x}}, \tag{4.11}$$

$$\bar{w}_t = \left(v_{\bar{x}} - \frac{v^2}{2} + \frac{3}{2}\bar{w}^2\right)_{\bar{x}}. \tag{4.12}$$

This system is equivalent to the Kaup or classical Boussinesq system. In fact it could be written as a single equation if we set

$$\bar{w} = p_{\bar{x}}.$$

One can now remove v from (4.11-12) and the result is a sort of modified Boussinesq equation

$$p_{tt} - p_{\bar{x}\bar{x}\bar{x}\bar{x}} - 4p_{\bar{x}}p_{\bar{x}t} + 6p_{\bar{x}}^2 p_{\bar{x}\bar{x}} - 2p_t p_{\bar{x}\bar{x}} = 0. \tag{4.13}$$

Unlike the AKNS case in which the Singular Manifold equations were also AKNS, for NLBq the Singular Manifold equations (4.11-12) are not the same system but rather they become the Kaup system (KS). As we will see in the next section this means that both systems NLBq and KS are related by a Miura transformation.

4.2 Lax pairs

The question of the linearization of the singular manifold equations is now a little bit more complicated that it was for AKNS. If we look for the dominant terms of (4.11-12)

$$v \sim v_0 \chi^a,$$

$$\bar{w} \sim \bar{w}_0 \chi^b.$$

The result is:

$$a = -1, \qquad b = -1, \qquad v_0 = \chi_{\bar{x}}, \qquad \bar{w}_0 = \pm\chi_{\bar{x}},$$

the $\pm$ sign confirms the well known fact that the Kaup system has two Painlevé branches [9], [8]. In the previous section we have explained that for those types of systems it

is necessary the introduction of two Singular Manifolds. From that point of view, the truncation ansatz for v and $\bar{w}$ is now:

$$v = \frac{{\psi^+}_{\bar{x}}}{\psi^+} + \frac{{\psi^-}_{\bar{x}}}{\psi^-} \Longrightarrow \phi_{\bar{x}} = \psi^+\psi^-, \tag{4.14}$$

$$\bar{w} = \frac{{\psi^+}_{\bar{x}}}{\psi^+} - \frac{{\psi^-}_{\bar{x}}}{\psi^-} \tag{4.15}$$

or

$$2\frac{{\psi^+}_x}{\psi^+} = v + w + 2\lambda, \tag{4.16}$$

$$2\frac{{\psi^-}_x}{\psi^-} = v - w - 2\lambda. \tag{4.17}$$

With this ansatz the expressions (4.5-6) for the truncated solutions (see Appendix D) can be linearized as:

$$0 = 2M_x(\psi^+_{xx} + M_x\psi^+) - (M_t + M_{xx} + 2\lambda M_x)\psi^+_x, \tag{4.18}$$

$$0 = \psi^+_t - \psi^+_{xx} + 2\lambda\psi^+_x - 2M_x\psi^+, \tag{4.19}$$

$$0 = 2M_x(\psi^-_{xx} + M_x\psi^-) + (M_t - M_{xx} + 2\lambda M_x)\psi^-_x, \tag{4.20}$$

$$0 = \psi^-_t + \psi^-_{xx} + 2\lambda\psi^-_x + 2M_x\psi^-. \tag{4.21}$$

To summarize, the existence of two branches in the Singular Manifold equations implies the existence of two classes of eigenfunctions of two different Lax pairs (4.18-19) and (4.20-21). This result reflects the fact that the equation is invariant under the discrete symmetry

$$\begin{aligned} x &\longrightarrow -x, \\ t &\longrightarrow -t, \\ M &\longrightarrow -M \end{aligned}$$

that transforms the first of the Lax pairs into the second one.

4.3 Darboux transformations

The generation of the Darboux transformations can be done in a similar way as we did in Section 2. If we write the Lax pairs for the iterated solution M'

$$0 = 2M'_x(\psi'^+_{2xx} + M'_x\psi'^+_2) - (M'_t + M'_{xx} + 2\lambda_2 M'_x)\psi'^+_{2x}, \tag{4.22}$$

$$0 = \psi'^+_{2t} - \psi'^+_{2xx} + 2\lambda_2\psi'^+_{2x} - 2M'_x\psi'^+_2, \tag{4.23}$$

$$0 = 2M'_x(\psi'^-_{2xx} + M'_x\psi'^-_2) + (M'_t - M'_{xx} + 2\lambda_2 M'_x)\psi'^-_{2x}, \tag{4.24}$$

$$0 = \psi'^-_{2t} + \psi'^-_{2xx} + 2\lambda_2\psi'^-_{2x} + 2M'_x\psi'^-_2, \tag{4.25}$$

where $\psi_2^{'+}, \psi_2^{'-}$ could be related to a singular manifold ϕ_2' in the form

$$\phi_2' = \psi_2^{'+}\psi_2^{'-}. \tag{4.26}$$

The consideration of (4.22-25) as coupled nonlinear equations allows us to write truncated expansions for $M', \psi_2^{'+}, \psi_2^{'-}, \phi_2'$.

$$M' = M + \frac{\phi_{1x}}{\phi_1}, \tag{4.27}$$

$$\psi_2^{'+} = \psi_2^+ + \frac{\Theta^+}{\phi_1}, \tag{4.28}$$

$$\psi_2^{'-} = \psi_2^- + \frac{\Theta^-}{\phi_1}, \tag{4.29}$$

$$\phi_2^{'} = \phi_2 + \frac{\Delta}{\phi_1}, \tag{4.30}$$

where

$$0 = 2M_x(\psi_{2xx}^+ + M_x\psi_2^+) - (M_t + M_{xx} + 2\lambda_2 M_x)\psi_{2x}^+, \tag{4.31}$$

$$0 = \psi_{2t}^+ - \psi_{2xx}^+ + 2\lambda_2\psi_{2x}^+ - 2M_x\psi_2^+, \tag{4.32}$$

$$0 = 2M_x(\psi_{2xx}^- + M_x\psi_2^-) + (M_t - M_{xx} + 2\lambda_2 M_x)\psi_{2x}^-, \tag{4.33}$$

$$0 = \psi_{2t}^- + \psi_{2xx}^- + 2\lambda_2\psi_{2x}^- + 2M_x\psi_2^-, \tag{4.34}$$

$$\phi_{2x} = \psi_2^+\psi_2^-, \tag{4.35}$$

$$0 = 2M_x(\psi_{1xx}^+ + M_x\psi_1^+) - (M_t + M_{xx} + 2\lambda_1 M_x)\psi_{1x}^+, \tag{4.36}$$

$$0 = \psi_{1t}^+ - \psi_{1xx}^+ + 2\lambda_1\psi_{1x}^+ - 2M_x\psi_1^+, \tag{4.37}$$

$$0 = 2M_x(\psi_{1xx}^- + M_x\psi_1^-) + (M_t - M_{xx} + 2\lambda_1 M_x)\psi_{1x}^-, \tag{4.38}$$

$$0 = \psi_{1t}^- + \psi_{1xx}^- + 2\lambda_1\psi_{1x}^- + 2M_x\psi_1^-, \tag{4.39}$$

$$\phi_{1x} = \psi_1^+\psi_1^-. \tag{4.40}$$

By substituting the truncation (4.27-30) in (4.22-26) it is possible to obtain

$$\Theta^+ = -\psi_1^+\Omega^+, \tag{4.41}$$

$$\Theta^- = -\psi_1^-\Omega^-, \tag{4.42}$$

$$\Delta = -\Omega^+\Omega^-, \tag{4.43}$$

where

$$\Omega^+ = \left(\frac{1}{\lambda_2 - \lambda_1}\right)\frac{\psi_1^-}{\psi_{1x}^+}(\psi_1^+\psi_{2x}^+ - \psi_2^+\psi_{1x}^+), \tag{4.44}$$

$$\Omega^- = \left(\frac{1}{\lambda_2 - \lambda_1}\right)\frac{\psi_2^-}{\psi_{2x}^+}(\psi_1^+\psi_{2x}^+ - \psi_2^+\psi_{1x}^+). \tag{4.45}$$

(4.27-30) together with (4.41-45) define Darboux transformations for NLBq.

4.4 Hirota's function

The generation of a new iteration can be done by using ϕ'_2 as the Singular Manifold for M' in order to construct

$$M'' = M' + \frac{\phi'_{2x}}{\phi'_2} = M + \frac{\tau_{12x}}{\tau_{12}}, \tag{4.46}$$

where

$$\tau_{12} = \phi'_2 \phi_1 \tag{4.47}$$

and by using (4.30) and (4.43)

$$\tau_{12} = \phi_2 \phi_1 - \Omega^+ \Omega^-. \tag{4.48}$$

4.5 Solitonic Solutions

The easiest nontrivial solutions can be obtained from the seminal solution

$$M = a_0 x. \tag{4.49}$$

For this solution, exponential solutions of (4.22-25) are

$$\psi_i^+ = \exp\{a_i(x - a_i t)\}, \qquad \psi_i^- = \exp\left\{-\frac{a_0}{a_i}\left(x - \frac{a_0}{a_i} t\right)\right\}, \tag{4.50}$$

where a_i are related with the spectral parameter in the form

$$\lambda_i = a_i + \frac{a_0}{a_i} \tag{4.51}$$

and the corresponding singular manifolds are

$$\phi_i = \frac{a_i}{a_i^2 - a_0}\,(\alpha_i + \psi_i^+ \psi_i^-), \tag{4.52}$$

where α_i are arbitrary constants. (4.44-45) gives:

$$\Omega^+ = \frac{a_2}{a_2 a_1 - a_0} \psi_1^- \psi_2^+, \qquad \Omega^- = \frac{a_1}{a_2 a_1 - a_0} \psi_1^+ \psi_2^-, \tag{4.53}$$

and (4.48) is:

$$\tau_{12} = \frac{a_1 a_2}{(a_1^2 - a_0)(a_2^2 - a_0)}\,(\alpha_1 + \psi_1^+ \psi_1^-)(\alpha_2 + \psi_2^+ \psi_2^-) - \frac{a_1 a_2}{(a_2 a_1 - a_0)^2} \psi_1^- \psi_2^+ \psi_1^+ \psi_2^-. \tag{4.54}$$

We can write the first (Fig. 3) and second iteration (Fig. 4) as:

$$M' = a_0 x + \frac{\phi_{1x}}{\phi_1}, \tag{4.55}$$

$$M'' = a_0 x + \frac{\tau_{12x}}{\tau_{12}}, \tag{4.56}$$

where

$$\phi_1 = \frac{\alpha_1 a_1}{a_1^2 - a_0}(1 + F_1), \tag{4.57}$$

$$\tau_{12} = \frac{a_1 a_2 \alpha_1 \alpha_2}{(a_1^2 - a_0)(a_2^2 - a_0)} \left\{1 + F_1 + F_2 + A_{12} F_1 F_2\right\}, \tag{4.58}$$

$$\alpha_i = \exp\left\{\left(a_i - \frac{a_0}{a_i}\right) x_{0i}\right\},$$

$$F_i = \frac{\psi_i^+ \psi_i^-}{\alpha_i} = \exp\left\{\left(a_i - \frac{a_0}{a_i}\right)\left[x - \left(a_i + \frac{a_0}{a_i}\right) t - x_{0i}\right]\right\}, \tag{4.59}$$

$$A_{12} = a_0 \left(\frac{a_2 - a_1}{a_1 a_2 - a_0}\right)^2. \tag{4.60}$$

5 The Kaup system. Miura transformation to NLBq

In the previous section, we have seen how the Kaup system (KS) arises as the Singular Manifold equation for NLBq. That suggests a Miura transformation between KS and NLBq [26], [22]. Let us write the Kaup system in the form

$$u_t = \eta_{xx} + 2uu_x, \tag{5.1}$$

$$\eta_t = u_{xx} + 2u\eta_x. \tag{5.2}$$

Note that if we set $u = p_x$, (5.1-2) can be expressed as follows

$$p_{tt} - p_{xxxx} - 4p_x p_{xt} + 6p_x^2 p_{xx} - 2p_t p_{xx} = 0. \tag{5.3}$$

This is precisely the Singular Manifold equation (4.13) for NLBq.

If we use Painlevé series for u and η

$$u = \sum_{j=0}^{\infty} u_j \chi^{j-a},$$

$$\eta = \sum_{j=0}^{\infty} \eta_j \chi^{j-b}$$

the dominant terms yield: $a = b = 1$ and

$$u_0 = \pm\chi_x, \tag{5.4}$$

$$\eta_0 = \chi_x. \tag{5.5}$$

The existence of two branches in the Painlevé expansion suggests the following change of functions:

$$u = m - \hat{m}, \tag{5.6}$$

$$\eta = m + \hat{m}. \tag{5.7}$$

With this change the addition and subtraction of (5.1) and (5.2) yields

$$m_t = m_{xx} + 2(m - \hat{m})m_x, \tag{5.8}$$

$$\hat{m}_t = -\hat{m}_{xx} + 2(m - \hat{m})\hat{m}_x. \tag{5.9}$$

5.1 Miura transformation

From (5.8) we can obtain $\hat{m}$. By substituting it in (5.9), the result is:

$$m_x^2(m_{tt} - m_{xxxx}) = 4m_x^3 m_{xx} + 2m_x(m_t m_{xx} - m_{xx} m_{xxx}) - m_{xx}(m_t^2 - m_{xx}^2) \quad (5.10)$$

that is precisely the (4.3) NLBq. In a similar way m can be obtained from (5.9). Its substitution in (5.8) takes the form

$$\hat{m}_x^2(\hat{m}_{tt} - \hat{m}_{xxxx}) = 4\hat{m}_x^3 \hat{m}_{xx} + 2\hat{m}_x(\hat{m}_t \hat{m}_{xx} - \hat{m}_{xx} \hat{m}_{xxx}) - \hat{m}_{xx}(\hat{m}_t^2 - \hat{m}_{xx}^2) \quad (5.11)$$

that is again NLBq. In consequence, the splitting (5.6-7) leads to the possibility of constructing Soliton Solutions of KS by linear superposition of two solutions of NLBq. Actually the inversion of (5.6-7) can be written as:

$$2m_x = u_x - u^2 + \partial_x^{-1} u_t, \quad (5.12)$$

$$2\hat{m}_x = -u_x - u^2 + \partial_x^{-1} u_t \quad (5.13)$$

that is a Miura transformation between KS and NLBq [26].

5.2 Bäcklund transformations

Although two solutions m and $\hat{m}$ of NLBq can be used to construct (by means of (5.6-7)) a solution of KS, these solutions are certainly not unrelated. Actually (5.8-9) establishes the correspondent relation between m and $\hat{m}$. This relationship can be written in the form:

$$m = \hat{m} + \frac{\hat{m}_t + \hat{m}_{xx}}{2\hat{m}_x}, \quad (5.14)$$

$$\hat{m} = m + \frac{m_{xx} - m_t}{2m_x} \quad (5.15)$$

which obviously is the Bäcklund transformation that relates the two solution of NLBq.

5.3 Two Singular Manifolds

The Singular Manifold approach derived in the previous section can be applied to m and $\hat{m}$. The Painlevé expansion takes the form

$$m' = m + \frac{\phi_x}{\phi} \Longrightarrow u' = u + \frac{\phi_x}{\phi} - \frac{\hat{\phi}_x}{\hat{\phi}}, \quad (5.16)$$

$$\hat{m}' = \hat{m} + \frac{\hat{\phi}_x}{\hat{\phi}} \Longrightarrow \eta' = \eta + \frac{\phi_x}{\phi} + \frac{\hat{\phi}_x}{\hat{\phi}}. \quad (5.17)$$

Noneless the Bäcklund transformations (5.14-15) imply that ϕ and $\hat{\phi}$ are not unrelated. The substitution of the Painlevé expansions (5.16-17) (or alternatively in (5.1-2)) gives rise to (see Appendix E) the coupling condition

$$\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}} = A\frac{\phi_x}{\phi} + \hat{A}\frac{\hat{\phi}_x}{\hat{\phi}}, \quad (5.18)$$

where

$$A = \frac{v-w}{2} + u, \tag{5.19}$$

$$\hat{A} = \frac{\hat{v}+\hat{w}}{2} - u, \tag{5.20}$$

and

$$\lambda = \hat{\lambda}, \tag{5.21}$$

$$u = \lambda + \frac{\hat{v}+\hat{w}-v+w}{2}. \tag{5.22}$$

By using the definitions (4.16-17), the expressions (5.19-22) are:

$$A = u + \lambda + \frac{\psi_x^-}{\psi^-}, \tag{5.23}$$

$$\hat{A} = -(u+\lambda) + \frac{\hat{\psi}_x^+}{\hat{\psi}^+}, \tag{5.24}$$

$$u + \lambda = \frac{\hat{\psi}_x^+}{\hat{\psi}^+} - \frac{\psi_x^-}{\psi^-}. \tag{5.25}$$

5.4 Lax pair for KS

The derivative of the coupling condition (5.18) with respect to x is:

$$A_x = A(\hat{v} - A - \hat{A}) = \frac{\hat{v}-\hat{w}-v+w}{2} = \frac{\hat{\psi}_x^-}{\hat{\psi}^-} - \frac{\psi_x^-}{\psi^-}, \tag{5.26}$$

$$\hat{A}_x = \hat{A}(v - A - \hat{A}) = \frac{v+w-\hat{v}-\hat{w}}{2} = \frac{\psi_x^+}{\psi^+} - \frac{\hat{\psi}_x^+}{\hat{\psi}^+}. \tag{5.27}$$

These expressions can be easily integrated as:

$$A = a\frac{\hat{\psi}^-}{\psi^-}, \tag{5.28}$$

$$\hat{A} = b\frac{\psi^+}{\hat{\psi}^+}. \tag{5.29}$$

Combined with (5.23-24) these formulae yield

$$\psi_x^- = a\hat{\psi}^- - (u+\lambda)\psi^-, \tag{5.30}$$

$$\hat{\psi}_x^+ = \hat{a}\psi^+ + (u+\lambda)\hat{\psi}^+. \tag{5.31}$$

The substitution of (5.25) in (5.30-31) leads to

$$a\hat{\psi}_x^- = (u_x - \eta_x)\psi^-, \tag{5.32}$$

$$\hat{a}\psi_x^+ = -(u_x + \eta_x)\hat{\psi}^+, \tag{5.33}$$

where we have used $\hat{\psi}_x^+\hat{\psi}_x^- = -\hat{m}_x\hat{\psi}^+\hat{\psi}^-$, $\psi_x^+\psi_x^- = -m_x\psi^+\psi^-$ (see Appendix D).

Those expressions can be written as

$$\begin{pmatrix} \psi^- \\ \hat{\psi}^- \end{pmatrix}_x = \begin{pmatrix} -(u+\lambda) & a \\ \dfrac{u_x - \eta_x}{a} & 0 \end{pmatrix} \begin{pmatrix} \psi^- \\ \hat{\psi}^- \end{pmatrix}, \tag{5.34}$$

$$\begin{pmatrix} \psi^+ \\ \hat{\psi}^+ \end{pmatrix}_x = \begin{pmatrix} 0 & \dfrac{-(u_x + \eta_x)}{\hat{a}} \\ \hat{a} & (u+\lambda) \end{pmatrix} \begin{pmatrix} \psi^+ \\ \hat{\psi}^+ \end{pmatrix}. \tag{5.35}$$

(5.34-35) are the spatial part of two components Lax pair for KS. The temporal part can be obtained from (4.17) and (4.19).

$$\begin{pmatrix} \psi^- \\ \hat{\psi}^- \end{pmatrix}_t = \begin{pmatrix} \left(\dfrac{1}{2a}\right)[\eta_{xx} - u_{xx} - (u-\lambda)(\eta_x - u_x)], & \dfrac{u_x - \eta_x}{2} \\ -\left(\dfrac{\eta_x + u_x}{2} + u^2 - \lambda^2\right), & a(u-\lambda) \end{pmatrix} \begin{pmatrix} \psi^- \\ \hat{\psi}^- \end{pmatrix}, \tag{5.36}$$

$$\begin{pmatrix} \psi^+ \\ \hat{\psi}^+ \end{pmatrix}_t = \begin{pmatrix} \dfrac{u_x + \eta_x}{2}, & \left(\dfrac{1}{2\hat{a}}\right)[-\eta_{xx} - u_{xx} - (u-\lambda)(\eta_x + u_x)] \\ \hat{a}(u-\lambda), & \left(\dfrac{\eta_x - u_x}{2} + u^2 - \lambda^2\right) \end{pmatrix} \begin{pmatrix} \psi^+ \\ \hat{\psi}^+ \end{pmatrix}. \tag{5.37}$$

6 Conclusions

This paper has been dealing all along with the Painlevé analysis in the version formulated by Weiss, Tabor and Carnevale [35]. Even though there is no rigorous proof so far available of the connection between the Painlevé property and integrability the work hereby reviewed aims to contribute to a better understanding of the validity and usefulness of methods based on the Painlevé property for studying Nonlinear Partial Differential Equations. With this idea in mind we would like to underscore some of the results that we have obtained in this paper by applying the Singular Manifold Method of Weiss [36], [37].

- In Section 2, we have applied the SMM to AKNS. This method have been proved to be quite useful to construct the Lax pair of AKNS. By applying the SMM to the Lax pair itself Darboux transformations and Hirota functions have been constructed algorithmically. The use of the SMM to construct solutions iteratively has been shown with the help of examples.
- A similar procedure has been used in Section 3 to study NLBq.
- The identification of Miura transformations and Bäcklund transformations by means of the SMM for equations with two Painlevé branches appears as the main goal of Sections 3 and 5. There sinh-Gordon and Kaup systems are presented as the modified versions of AKNS and NLBq respectively. Two component-Lax pairs for both systems are obtained from the AKNS and NLBq Lax pairs as induced by the Miura map.

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A Appendix

By substitution of (2.3) in (2.1) we obtain a polynomial in $\left(\frac{\phi_x}{\phi}\right)$ whose coefficients are (we have used the code MAPLE V for the algebraic computer algebra):

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)^3$

$$4M_y + 2v_y + 8qM_x + qv^2 + 2qv_x = 0. \tag{A.1}$$

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)^2$

$$\begin{aligned}&-4M_{xy} - 2qM_{xx} - 6vM_y - 12qvM_x - 8q_xM_x - 2q_xv_x\\ &\quad -2q_xv^2 - \frac{7}{2}vv_xq - \frac{1}{2}qv_{xx} - 3vv_y - \frac{3}{2}v^3 = 0.\end{aligned} \tag{A.2}$$

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)$

$$\begin{aligned}&8vM_{xy} + 4M_{xx}(q_x + qv) + 4M_y(v_x + v^2) + 8M_x(qv^2 + vq_x + v_y) + v_{yxx}\\ &+3vv_{xy} + v_{xx}(q_x + qv) + 3v_y(v_x + v^2) + 3qv^2v_x + 3vv_xq_x + v^3q_x + qv^4 = 0.\end{aligned} \tag{A.3}$$

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)^0$

$$0 = M_{yxxx} + 4M_yM_{xx} + 8M_xM_{xy}. \tag{A.4}$$

(A.4) means that M is a solution of AKNS and (A.1) can be used to obtain

$$M_y = -2qM_x - \frac{1}{2}v_y - \frac{1}{4}qv^2 - \frac{1}{2}qv_x. \tag{A.5}$$

The substitution of (A.5) in (A.2) is

$$M_{xx} + \frac{v_{xx}}{4} + \frac{vv_x}{4} = 0$$

that can be integrated as

$$M_x = -\frac{v_x}{4} - \frac{v^2}{8} - \frac{\lambda(t)}{2}, \tag{A.6}$$

where λ is a constant for the integration with respect to x. The substitution of (A.6) in (A.5) is:

$$M_y = -\frac{v_y}{2} + \lambda(t)q. \tag{A.7}$$

The cross derivatives of (A.6) and (A.7) yield

$$v_{xy} - vv_y = 4\lambda q_x,$$

$$\frac{d\lambda}{dt} = 0$$

that are the Singular Manifold equations.

B Appendix

- Let us substitute (3.22) in (3.3). The result is:

$$\frac{\phi_x}{\phi}(v+2u) + \frac{\hat{\phi}_x}{\hat{\phi}}(\hat{v}-2u) - 2\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}} = 0 \tag{B.1}$$

that compared with (3.24) reads

$$A = \frac{v}{2} + u, \tag{B.2}$$

$$\hat{A} = \frac{\hat{v}}{2} - u. \tag{B.3}$$

- Taking the derivative of (3.24) with respect to x.

C Appendix

The substitution of (4.4) in (4.2) leads to a polynomial in $\frac{\phi_x}{\phi}$ whose coefficients are:

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)^3$

$$4M_{xx} - w_t + v_{xx} + ww_x + vv_x = 0. \tag{C.1}$$

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)^2$

$$\begin{aligned} &N_t - M_{xxx} + 6vM_{xx} - 2M_t w - 6M_x^2 + M_x(w^2 - v^2 - 4v_x) \\ &\quad + w_x^2 - v_x^2 + vv_{xx} + v^2 v_x + vww_x - vw_t = 0. \end{aligned} \tag{C.2}$$

- Coefficient in $\left(\frac{\phi_x}{\phi}\right)$

$$\begin{aligned} &vN_t - vM_{xxx} + 2M_{xx}(v_x + v^2) - 2M_t(w_x + wv) - 6vM_x^2 \\ &\quad + M_x(-v_{xx} - v^3 + w_t - 3vv_x + ww_x + w^2 v) = 0. \end{aligned} \tag{C.3}$$

If we set $w_t = \left(v_x - \frac{v^2}{2} + p\right)_x$, equation (C.1) can be integrated in x as:

$$M_x = \frac{1}{4}\left(p - v^2 - \frac{w^2}{2} + 2\lambda^2(t)\right), \tag{C.4}$$

where λ is an integration constant. By multiplying (C.3) for v and subtracting (C.2)

$$2w_x(-2M_t - vww_x + 2wM_x) + (v_x + 2M_x)(4M_{xx} - 2vv_x) = 0. \tag{C.5}$$

The substitution of (C.4) in (C.5) yields

$$M_t = \frac{1}{8}\left[-4vw_x + 2(z-w)v_x + (z+w)\left(p - v^2 - \frac{w^2}{2} + 2\lambda^2(t)\right)\right], \tag{C.6}$$

where we have equated

$$p_x = zw_x. \tag{C.7}$$

The compatibility $M_{xt} = M_{tx}$ between (C.4) and (C.7) implies that

$$z = 3w + 4\lambda, \tag{C.8}$$

$$p = \frac{3}{2}w^2 + 4\lambda w + 2\lambda^2, \tag{C.9}$$

$$\frac{d\lambda}{dt} = 0,$$

and therefore we obtain

$$w_t = \left(v_x - \frac{v^2}{2} + \frac{3}{2}w^2 + 4\lambda w + 2\lambda^2\right)_x, \tag{C.10}$$

$$M_x = \frac{1}{4}\left[(w+2\lambda)^2 - v^2\right], \tag{C.11}$$

$$M_t = \frac{1}{2}\left\{(w+2\lambda)v_x - vw_x + (w+\lambda)[(w+2\lambda)^2 - v^2]\right\}. \tag{C.12}$$

D Appendix

To simplify the calculation let us define (see (4.16-17))

$$2\alpha^+ = 2\frac{\psi_x^+}{\psi^+} = v + w + 2\lambda, \tag{D.1}$$

$$2\alpha^- = 2\frac{\psi_x^-}{\psi^-} = v - w - 2\lambda, \tag{D.2}$$

or

$$v = \alpha^+ + \alpha^-, \tag{D.3}$$

$$w + \lambda = \alpha^+ - \alpha^-. \tag{D.4}$$

The substitution of (D.3-4) in (4.5-6) is

$$M_x = -\alpha^+\alpha^-, \tag{D.5}$$

$$M_t = -\alpha^+\alpha^- \left(2\alpha^+ - 2\alpha^- - 2\lambda + \frac{\alpha_x^+}{\alpha^+} - \frac{\alpha_x^-}{\alpha^-}\right). \tag{D.6}$$

• In order to remove α^- from (D.5) and (D.6), we use (D.5) to set

$$\alpha^- = -\frac{M_x}{\alpha^+}. \tag{D.7}$$

Its substitution in (D.6) provides:

$$\begin{aligned} M_t &= M_x \left[-\frac{M_{xx}}{M_x} + 2\frac{M_x}{\alpha^+} + 2\alpha^+ + 2\frac{\alpha_x^+}{\alpha^+} - 2\lambda\right] \\ &= M_x \left[-\frac{M_{xx}}{M_x} + 2M_x\frac{\psi^+}{\psi_x^+} + 2\frac{\psi_{xx}^+}{\psi_x^+} - 2\lambda\right] \end{aligned} \tag{D.8}$$

that is (4.18). (4.20) can be obtained in the same form by removing α^+ between (D.5) and (D.6).

• The temporal part of the Lax pair is obtained from the derivation of (D.1) with respect to t

$$2\alpha_t^+ = 2\left(\frac{\psi_x^+}{\psi^+}\right)_t = v_t + w_t. \tag{D.9}$$

The use of (4.9-10) yields:

$$2\alpha_t^+ = \left[w_x + wv + v_x - \frac{v^2}{2} + \frac{3}{2}(w + 2\lambda)^2 - 2\lambda(w + 2\lambda)\right]_x \tag{D.10}$$

that with the use of (D.3) and (D.4) finally leads to

$$\alpha_t^+ = \left[\alpha_x^+ + \alpha^{+2} - 2\lambda\alpha^+ - 4\alpha^+\alpha^-\right]_x. \tag{D.11}$$

Removing α^- with the aid of (D.7)

$$\alpha_t^+ = \left[\alpha_x^+ + \alpha^{+2} - 2\lambda\alpha^+ - 4\alpha^+ 2M_x\right]_x. \tag{D.12}$$

Finally we can substitute (D.1) and integrate out in x as

$$\frac{\psi_t^+}{\psi^+} = \frac{\psi_{xx}^+}{\psi^+} - 2\lambda\frac{\psi_x^+}{\psi^+} + 2M_x \tag{D.13}$$

that is precisely (4.19). The expression (4.21) can be obtained by repeating the same process with (D.2).

E Appendix

- The substitution of (5.16) in (5.1) gives

$$\frac{\phi_x}{\phi}(-w+v+2u)+\frac{\hat{\phi}_x}{\hat{\phi}}(\hat{w}+\hat{v}-2u)-2\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}}. \tag{E.1}$$

The comparison with (5.18) yields to

$$A=u+\frac{v-w}{2}, \tag{E.2}$$

$$\hat{A}=-u+\frac{\hat{v}+\hat{w}}{2}. \tag{E.3}$$

- The derivative of (5.18) with respect to x provides just like in Appendix B

$$A_x=A(\hat{v}-A-\hat{A}), \tag{E.4}$$

$$\hat{A}_x=\hat{A}(v-A-\hat{A}). \tag{E.5}$$

- The substitution of (5.16) in (5.2) gives us:

$$\begin{aligned}&\frac{\phi_x}{\phi}[-w_x-wv+v_x+v^2+2uv+2\eta_x]+\frac{\hat{\phi}_x}{\hat{\phi}}[-\hat{w}_x-\hat{w}\hat{v}-\hat{v}_x-\hat{v}^2+2u\hat{v}-2\eta_x]\\&\quad+\left(\frac{\phi_x}{\phi}\right)^2[w-v-2u+2A]+\left(\frac{\hat{\phi}_x}{\hat{\phi}}\right)^2[\hat{w}+\hat{v}-2u-2\hat{A}]\\&\quad+2\frac{\phi_x}{\phi}\frac{\hat{\phi}_x}{\hat{\phi}}\left[\hat{v}-v+\frac{\phi_x}{\phi}-\frac{\hat{\phi}_x}{\hat{\phi}}\right]=0.\end{aligned} \tag{E.6}$$

The use of (E.4-5) and (5.18) yields to:

$$\begin{aligned}&\frac{\phi_x}{\phi}[-w_x-wv+v_x+v^2+2uv+2\eta_x+2A(\hat{v}-v+\hat{A}-A)]\\&\quad+\frac{\hat{\phi}_x}{\hat{\phi}}[-\hat{w}_x-\hat{w}\hat{v}-\hat{v}_x-\hat{v}^2+2u\hat{v}-2\eta_x+2\hat{A}(\hat{v}-v+\hat{A}-A)]=0.\end{aligned}$$

Setting to 0 both coefficients and using (E.2-5)

$$2\hat{m}_x=\eta_x-u_x=-2A(\hat{v}-A)=0, \tag{E.7}$$

$$2m_x=\eta_x+u_x=-2\hat{A}(v-\hat{A})=0. \tag{E.8}$$

The comparison between (E.8-9) and (4.5) leads to

$$A=\frac{\hat{v}}{2}\pm\frac{\hat{w}+2\hat{\lambda}}{2}, \tag{E.9}$$

$$\hat{A}=\frac{v}{2}\pm\frac{w+2\lambda}{2} \tag{E.10}$$

that compared with (E.2-3) means that A requires the plus sign and $\hat{A}$ the minus sign and

$$\lambda=\hat{\lambda} \tag{E.11}$$

which means that

$$u=A+\frac{w-v}{2}=-\hat{A}+\frac{\hat{v}+\hat{w}}{2}=\frac{1}{2}[w-v+\hat{w}+\hat{v}+2\lambda]. \tag{E.12}$$

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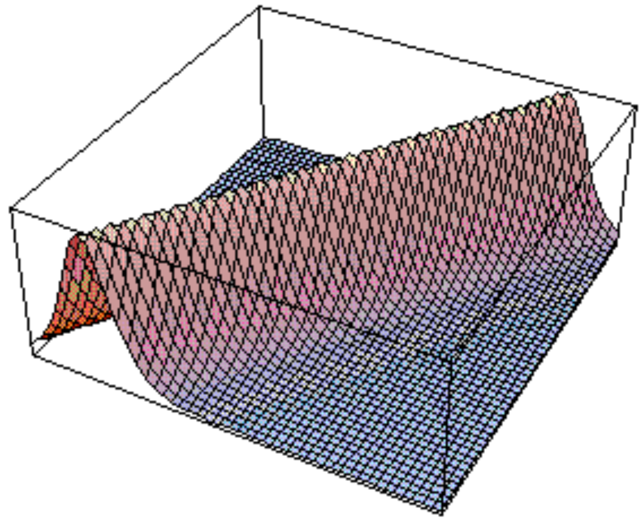

figure 1

One soliton solution M'_x *for* $k_1 = 0,2,\ k_2 = 0,3,\ a_0 = 0,144$

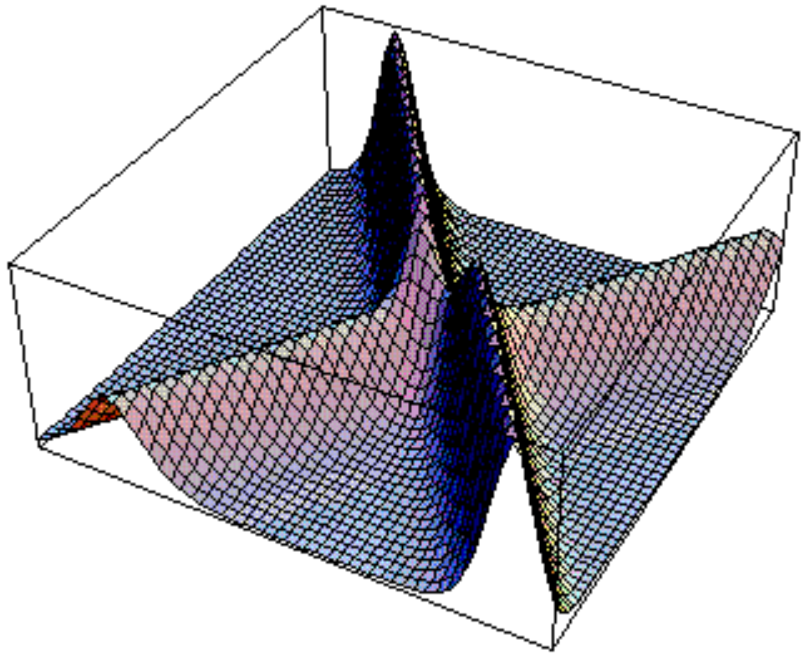

figure 2
Two soliton solution M''_x *for* $k_1 = 0,2,\ k_2 = 0,3,\ a_0 = 0,144$

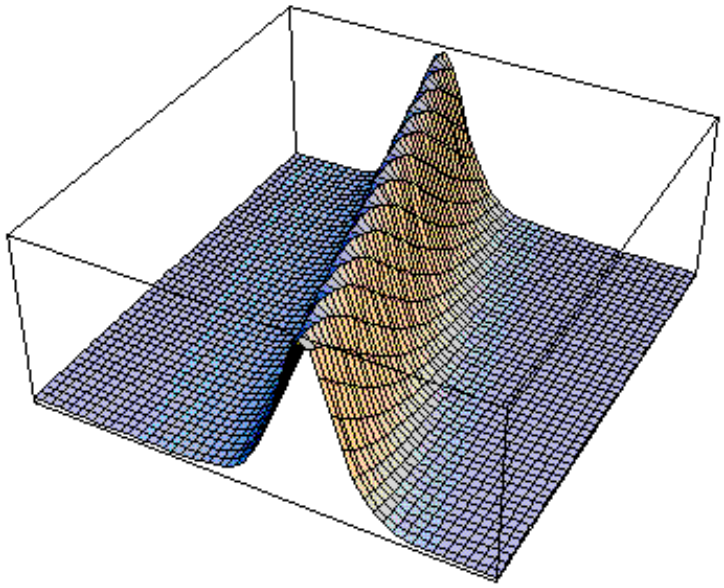

figure 3

One soliton solution M'_x *for* $a_1 = 1.2$ $a_2 = 1.4$, $a_0 = 3$

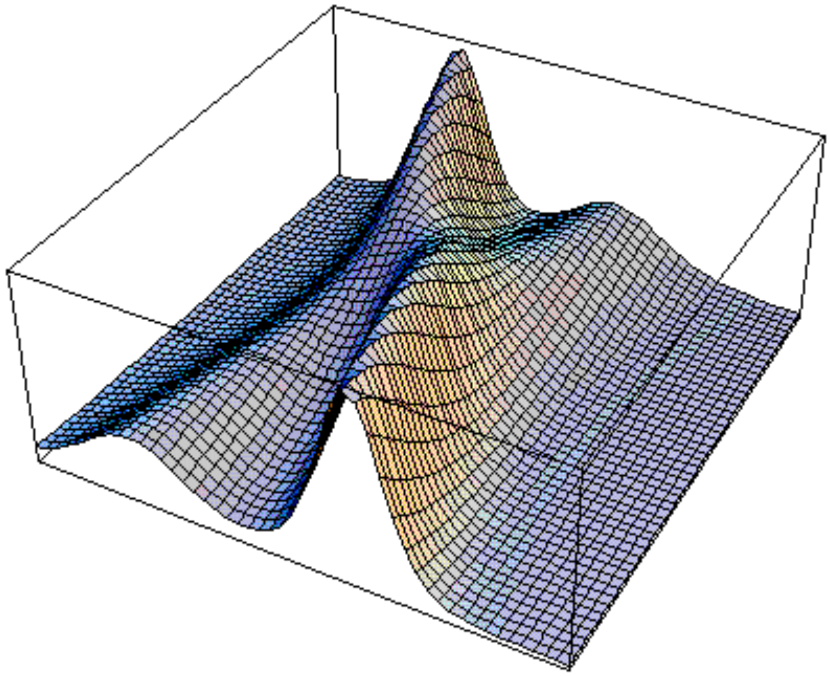

figure 4

Two soliton solution M_x'' *for* $a_1 = 1.2$ $a_2 = 1.4$, $a_0 = 3$